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2007 J. Phys. A: Math. Theor. 40 433

(http://iopscience.iop.org/1751-8121/40/3/007)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **40** (2007) 433–453 doi:10.1088/1751-8113/40/3/007

Discrete coupled derivative nonlinear Schrödinger equations and their quasi-periodic solutions

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Received 1 September 2006, in final form 23 November 2006
Published 20 December 2006
Online at stacks.iop.org/JPhysA/40/433

Abstract
A hierarchy of nonlinear differential-difference equations associated with a discrete isospectral problem is proposed, in which a typical differential-difference equation is a discrete coupled derivative nonlinear Schrödinger equation. With the help of the nonlinearization of the Lax pairs, the hierarchy of nonlinear differential-difference equations is decomposed into a new integrable symplectic map and a class of finite-dimensional integrable Hamiltonian systems. Based on the theory of algebraic curve, the Abel–Jacobi coordinates are introduced to straighten out the corresponding flows, from which quasi-periodic solutions for these differential-difference equations are obtained resorting to the Riemann-theta functions. Moreover, a (2+1)-dimensional discrete coupled derivative nonlinear Schrödinger equation is proposed and its quasi-periodic solutions are derived.

PACS numbers: 02.30.Ik, 02.30.Ks

1. Introduction

The derivative nonlinear Schrödinger (DNLS) equation

$$\mathrm{i}v_T + v_{XX} + 2\mathrm{i}(|v|^2 v)_X = 0 \tag{1.1}$$

arises in a wide variety of fields, such as plasma physics, electromagnetic waves in ferromagnetic, antiferromagnetic or dielectric systems [1–10]. Integrability of the DNLS equation was established in [11] with the derivative of one-soliton solution. And N-soliton formulae, periodic solutions and almost periodic solutions for the DNLS equation have also been obtained by using various approaches [12–17]. In [18], an integrable discretization of the coupled DNLS equation is proposed by constructing the Lax pair. The discrete coupled DNLS systems admit the reduction of complex conjugation between two dependent variables and possess bi-Hamiltonian structure.

In this paper, we propose a new integrable discretization of the coupled DNLS equation:

$$
\begin{aligned}
u(n)_t = \Delta\Bigg[& \frac{1}{1+u(n)v(n)}\Delta\frac{u(n)}{1+u(n)v(n)} \\
& - \frac{u(n)}{(1+u(n)v(n))^2}\left(\frac{u(n+1)v(n)}{1+u(n+1)v(n+1)} + \frac{u(n)v(n-1)}{1+u(n-1)v(n-1)}\right)\Bigg], \\
v(n)_t = \Delta^*\Bigg[& \frac{-1}{1+u(n)v(n)}\Delta^*\frac{v(n)}{1+u(n)v(n)} \\
& + \frac{v(n)}{(1+u(n)v(n))^2}\left(\frac{u(n+1)v(n)}{1+u(n+1)v(n+1)} + \frac{u(n)v(n-1)}{1+u(n-1)v(n-1)}\right)\Bigg],
\end{aligned} \tag{1.2}
$$

where the difference operator $\Delta f_n = f_{n+1} - f_n$ and its adjoint $\Delta^* f_n = f_{n-1} - f_n$. It is obvious that (1.2) do not admit a reduction of complex conjugation between the potentials. Usually, the system might exhibit singularities, solutions that blow up in finite time etc. The continuum limit of (1.2) is exactly the coupled DNLS equations [11]

$$
\mathrm{i}u_T - u_{XX} + 2\mathrm{i}(u^2v)_X = 0, \qquad \mathrm{i}v_T + v_{XX} + 2\mathrm{i}(uv^2)_X = 0, \tag{1.3}
$$

which are reduced to be the DNLS equation (1.1) as $u = v^*$. We first introduce a discrete spectral problem and derive the corresponding hierarchy of nonlinear differential-difference equations, in which a typical member is the discrete coupled DNLS equation (1.2). Based on the nonlinearization approach of the Lax pair [19–21], a new symplectic map and a class of new finite-dimensional Hamiltonian systems are derived, which are further proved to be completely integrable in the Liouville sense [22–24]. Based on the theory of algebraic curves, the straightening out of various flows is exactly given through the Abel–Jacobi coordinates. As an application, quasi-periodic solutions for the hierarchy of nonlinear differential-difference equations are obtained. Moreover, a (2+1)-dimensional discrete coupled DNLS equation is proposed with the aid of the first two nontrivial equations in this hierarchy

$$
\begin{aligned}
u(n)_t &= u(n)_{yy} - 2\Delta\frac{u(n)^2v(n-1)}{(1+u(n)v(n))^2(1+u(n-1)v(n-1))}, \\
v(n)_t &= -v(n)_{yy} + 2\Delta^*\frac{u(n+1)v(n)^2}{(1+u(n)v(n))^2(1+u(n+1)v(n+1))},
\end{aligned} \tag{1.4}
$$

whose quasi-periodic solutions of this equation is given in the meantime. It is shown that the continuous limit of (1.4) is a (2+1)-dimensional coupled DNLS equation

$$
\mathrm{i}u_T - u_{YY} + 2\mathrm{i}(u^2v)_X = 0, \tag{1.5}
$$

$$
\mathrm{i}v_T + v_{YY} + 2\mathrm{i}(v^2u)_X = 0, \tag{1.6}
$$

which is become into a (2+1)-dimensional DNLS equation as $u = v^*$:

$$
\mathrm{i}v_T + v_{YY} + 2\mathrm{i}(|v|^2v)_X = 0. \tag{1.7}
$$

The outline of the present paper is as follows. In section 2, we introduce a discrete spectral problem and derive the corresponding hierarchy of nonlinear differential-difference equations. It is proved that the continuous limit of the second member in the hierarchy is exactly the coupled DNLS equation. In section 3, we give a Bargmann constraint between the potentials and eigenfunctions, from which a new symplectic map and its involutive system of conserved integrals are obtained. In section 4, solutions of these nonlinear differential-difference equations are decomposed into solving compatible Hamiltonian systems of ordinary differential equations and the symplectic map. In section 5, with the help of elliptic coordinates and quasi-Abel–Jacobi coordinates, the integrability of the symplectic map and the resulting finite-dimensional Hamiltonian systems is proved rigorously. In sections 6 and 7, the Abel–Jacobi coordinates are introduced, by which the straightening out of the continuous flow and the discrete flow are studied in detail. In section 8, based on the Riemann–Jacobi inversion, quasi-periodic solutions for the hierarchy of nonlinear differential-difference equations are obtained by using the Riemann theta functions. Moreover, we propose a (2+1)-dimensional discrete coupled DNLS equation and derive its quasi-periodic solutions and the continuous limit.

2. Discrete nonlinear evolution equations

Let E be the shift operator defined by $Ef(n) = f(n+1)$, and $E^{-1}f(n) = f(n-1)$, $\Delta = E-1$, $\Delta^* = E^{-1}-1$. We first introduce a discrete spectral problem

$$E\chi(n) = U(n)\chi(n), \qquad U(n) = \frac{1}{\sqrt{1+\lambda}}\begin{pmatrix} 1+\lambda(1+u(n)v(n)) & \lambda u(n) \\ v(n) & 1 \end{pmatrix}, \tag{2.1}$$

which is a similar extension in [25]. Here λ is a constant spectral parameter, $u(n)$ and $v(n)$ are two potentials. In order to derive a hierarchy of discrete nonlinear evolution equations associated with (2.1), we introduce Lenard's gradient sequence $g_j(n) = (C_j(n+1), B_j(n), A_j(n))^T$ by the recursion equation

$$K_n g_{j-1}(n) = J_n g_j(n), \qquad J_n g_0(n) = 0, \qquad j \geqslant 0 \tag{2.2}$$

with the condition $g_j|_{(u(n),v(n))=0} = 0$, $(j \geqslant 1)$, where two operators K_n and J_n are defined as [23]

$$K_n = \begin{pmatrix} 0 & \Delta & 0 \\ -\Delta^* & 0 & 0 \\ -u(n) & v(n) & \Delta \end{pmatrix}, \qquad J_n = \begin{pmatrix} 0 & 1+u(n)v(n) & -u(n)(1+E) \\ -(1+u(n)v(n)) & 0 & v(n)(1+E) \\ -u(n) & v(n) & \Delta \end{pmatrix}.$$

The equation $J_n g_0(n) = 0$ has a special solution

$$g_0(n) = \left(\frac{v(n)}{1+u(n)v(n)}, \frac{u(n)}{1+u(n)v(n)}, \frac{1}{2}\right)^T, \tag{2.3}$$

which is taken as a starting point. Then $g_j(n)$ is uniquely determined by the recursion equation (2.2). It is easy to see that

$$g_1(n) = \begin{pmatrix} \frac{1}{1+u(n)v(n)}\Delta^*\frac{v(n)}{1+u(n)v(n)} - \frac{v(n)}{(1+u(n)v(n))^2}\left(\frac{u(n+1)v(n)}{1+u(n+1)v(n+1)} + \frac{u(n)v(n-1)}{1+u(n-1)v(n-1)}\right) \\ \frac{1}{1+u(n)v(n)}\Delta\frac{u(n)}{1+u(n)v(n)} - \frac{u(n)}{(1+u(n)v(n))^2}\left(\frac{u(n+1)v(n)}{1+u(n+1)v(n+1)} + \frac{u(n)v(n-1)}{1+u(n-1)v(n-1)}\right) \\ -\frac{u(n)v(n-1)}{(1+u(n)v(n))(1+u(n-1)v(n-1))} \end{pmatrix}. \tag{2.4}$$

Assume that $\chi(n)$ satisfies the discrete spectral problem (2.1) and the auxiliary problem

$$\chi(n)_{t_m} = V(n)^{(m)}\chi(n), \qquad V(n)^{(m)} = \begin{pmatrix} \lambda A_n^{(m)} & \lambda B_n^{(m)} \\ C_n^{(m)} & -\lambda A_n^{(m)} \end{pmatrix} \tag{2.5}$$

with

$$A_n^{(m)} = \sum_{j=0}^m A_j(n)\lambda^{m-j}, \qquad B_n^{(m)} = \sum_{j=0}^m B_j(n)\lambda^{m-j}, \qquad C_n^{(m)} = \sum_{j=0}^m C_j(n)\lambda^{m-j}.$$

Then the compatibility condition between (2.1) and (2.5) yields the discrete zero-curvature equation, $U(n)_{tm}+U(n)V(n)^{(m)}-V(n+1)^{(m)}U(n)=0$, which is equivalent to the hierarchy of discrete evolution equations

$$(u(n),v(n))^T_{t_m}=X_m(n),\qquad m\geqslant 0 \tag{2.6}$$

with

$$X_m(n)=\begin{pmatrix}0&\Delta\\-\Delta^*&0\end{pmatrix}\begin{pmatrix}C_m(n+1)\\B_m(n)\end{pmatrix}=\mathcal{P}K_n g_m(n)=\mathcal{P}J_n g_{m+1}(n),$$

where $\mathcal{P}$ is the projective map $\gamma=(\gamma^1,\gamma^2,\gamma^3)^T\rightarrow(\gamma^1,\gamma^2)^T$. The first two members in (2.6), $(t_1=t)$, are as follows:

$$u(n)_{t_0}=\Delta\frac{u(n)}{1+u(n)v(n)},\qquad v(n)_{t_0}=-\Delta^*\frac{v(n)}{1+u(n)v(n)} \tag{2.7}$$

and

$$\begin{aligned}u(n)_t&=\Delta\left[\frac{1}{1+u(n)v(n)}\Delta\frac{u(n)}{1+u(n)v(n)}\right.\\&\qquad\left.-\frac{u(n)}{(1+u(n)v(n))^2}\left(\frac{u(n+1)v(n)}{1+u(n+1)v(n+1)}+\frac{u(n)v(n-1)}{1+u(n-1)v(n-1)}\right)\right],\\v(n)_t&=\Delta^*\left[\frac{-1}{1+u(n)v(n)}\Delta^*\frac{v(n)}{1+u(n)v(n)}\right.\\&\qquad\left.+\frac{v(n)}{(1+u(n)v(n))^2}\left(\frac{u(n+1)v(n)}{1+u(n+1)v(n+1)}+\frac{u(n)v(n-1)}{1+u(n-1)v(n-1)}\right)\right],\end{aligned} \tag{2.8}$$

which is exactly (1.2). Here we the explicit expressions of $V(n)^{(m)}$ for the cases $m=0,1$:

$$V(n)^{(0)}=\begin{pmatrix}\frac{1}{2}\lambda&\lambda\frac{u(n)}{1+u(n)v(n)}\\\frac{v(n-1)}{1+u(n-1)v(n-1)}&-\frac{1}{2}\lambda\end{pmatrix},\qquad V(n)^{(1)}=\begin{pmatrix}\lambda A_n^{(1)}&\lambda B_n^{(1)}\\C_n^{(1)}&-\lambda A_n^{(1)}\end{pmatrix}, \tag{2.9}$$

which determine the temporal part of the Lax pair for (2.7) or (2.8), respectively, where

$$\begin{aligned}A_n^{(1)}&=\frac{1}{2}\lambda-\frac{u(n)v(n-1)}{(1+u(n)v(n))(1+u(n-1)v(n-1))},\\B_n^{(1)}&=\lambda\frac{u(n)}{1+u(n)v(n)}+\frac{1}{1+u(n)v(n)}\Delta\frac{u(n)}{1+u(n)v(n)}\\&\qquad-\frac{u(n)}{(1+u(n)v(n))^2}\left(\frac{u(n+1)v(n)}{1+u(n+1)v(n+1)}+\frac{u(n)v(n-1)}{1+u(n-1)v(n-1)}\right),\\C_n^{(1)}&=\lambda\frac{v(n-1)}{1+u(n-1)v(n-1)}+\frac{1}{1+u(n-1)v(n-1)}\Delta^*\frac{v(n-1)}{1+u(n-1)v(n-1)}\\&\qquad-\frac{v(n-1)}{(1+u(n-1)v(n-1))^2}\left(\frac{u(n)v(n-1)}{1+u(n)v(n)}+\frac{u(n-1)v(n-2)}{1+u(n-2)v(n-2)}\right).\end{aligned}$$

In what follows, we discuss the continuous limit of (2.8). In the same way similar to [26], the functions $u(n)$ and $v(n)$ on a lattice with a small step h are defined as

$$u(n)=\mathrm{i}u(x)h,\qquad v(n)=v(x)h,\qquad E^k(u)=\mathrm{i}u(x+kh)h,\qquad E^k(v)=v(x+kh)h. \tag{2.10}$$

Substituting the above expressions into equation (2.8) and expanding it in a power series in h, we arrive at

$$\begin{aligned} \mathrm{i}u_t &= \mathrm{i}u_{xx}h^2 + 4uu_x vh^3 + (2u^2 v_x + \mathrm{i}u_{xxx})h^3 + o(h^4), \\ v_t &= -v_{xx}h^2 - 4\mathrm{i}uvv_x h^3 - (2\mathrm{i}u_x v^2 - v_{xxx})h^3 + o(h^4). \end{aligned}$$

Under the transformation $T = \mathrm{i}h^4 t$, $X = hx$, we obtain by taking $h \to 0$ that the coupled DNLS equation

$$\mathrm{i}u_T - u_{XX} + 2\mathrm{i}(u^2 v)_X = 0, \qquad \mathrm{i}v_T + v_{XX} + 2\mathrm{i}(uv^2)_X = 0, \tag{2.11}$$

which is reduced to the DNLS equation (1.1) as $u = v^*$.

Let us introduce the generating function of $\{g_j(n)\}$

$$g_\lambda(n) = \sum_{j=0}^{\infty} g_j(n)\lambda^{-j-1}, \tag{2.12}$$

which satisfies

$$(K_n - \lambda J_n)g_\lambda(n) = 0. \tag{2.13}$$

Proposition 2.1. *Let $\sigma(u(n), v(n), \lambda)$ be a linear map defined by*

$$V(n) = \sigma(u(n), v(n), \lambda)[\gamma(n)] = \begin{pmatrix} \lambda\gamma^3(n) & \lambda\gamma^2(n) \\ E^{-1}\gamma^1(n+1) & -\lambda\gamma^3(n) \end{pmatrix}. \tag{2.14}$$

Then the discrete commutative relation

$$(EV(n))U(n) - U(n)V(n) = U_*(u(n), v(n), \lambda)(\mathcal{P}(K_n - \lambda J_n)\gamma(n)) \tag{2.15}$$

holds for any function $\gamma(n) = (\gamma^1(n+1), \gamma^2(n), \gamma^3(n))^T$, where

$$\begin{aligned} U_*(u(n), v(n), \lambda)\begin{pmatrix} \delta u(n) \\ \delta v(n) \end{pmatrix} &= \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0} U(u(n) + \varepsilon\delta u(n), v(n) + \varepsilon\delta v(n), \lambda) \\ &= \frac{1}{\sqrt{1+\lambda}}\begin{pmatrix} \lambda(v(n)\delta u(n) + u(n)\delta v(n)) & \lambda\delta u(n) \\ \delta v(n) & 0 \end{pmatrix}. \end{aligned}$$

Corollary 2.2. *Equation $(K_n - \lambda J_n)\gamma(n) = 0$ implies $\det\sigma[\gamma(n)] =$ constant independent of n.*

Proof. By (2.15), $(K_n - \lambda J_n)\gamma(n) = 0$ means that $V(n+1) = U(n)V(n)U(n)^{-1}$. Thus $\det V(n+1) = \det V(n)$. □

By corollary 2.2, $\det\sigma[g_\lambda] =$ constant implies that

$$\det\sigma[g_\lambda] = -\tfrac{1}{4} \tag{2.16}$$

with the aid of the initial condition in (2.2).

3. A new symplectic map

In this section, we shall discuss the nonlinearization of the discrete eigenvalue problem (2.1) to give an integrable symplectic map. Here and in what follows, we stipulate usually to write $f(n) = f$, $f(n+k) = E^k f$, $k \in \mathcal{Z}$, for the sake of simplicity. Assume that λ_k and

$\chi = (p_k, q_k)^T$, $(1 \leqslant k \leqslant N)$, are N distinct eigenvalues and the associated eigenfunctions for the discrete spectral problem (2.1), respectively. A direct calculation shows that

$$\begin{aligned} &-\Delta p_k^2 = -\lambda_k[(1+uv)p_k^2 + u(1+E)p_kq_k], \\ &\Delta^*\lambda_k(Eq_k)^2 = \lambda_k^2(1+uv)(Eq_k)^2 - \lambda_k v(1+E)p_kq_k, \\ &-u\lambda_k(Eq_k)^2 - vp_k^2 + \Delta p_kq_k = 0, \end{aligned} \tag{3.1}$$

which implies

$$(K - \lambda_k J)\widetilde{\nabla}\lambda_k = 0 \tag{3.2}$$

with

$$\widetilde{\nabla}\lambda_k = \begin{pmatrix} \nabla\lambda_k \\ p_kq_k \end{pmatrix}, \qquad \nabla\lambda_k = \begin{pmatrix} \frac{\delta\lambda_k}{\delta u} \\ \frac{\delta\lambda_k}{\delta v} \end{pmatrix} = \begin{pmatrix} \lambda_k(Eq_k)^2 \\ -p_k^2 \end{pmatrix}. \tag{3.3}$$

Let us consider N copies of spectral problem (2.1)

$$E\begin{pmatrix} p_k \\ q_k \end{pmatrix} = \frac{1}{\sqrt{1+\lambda_k}}\begin{pmatrix} 1+\lambda_k(1+uv) & \lambda_k u \\ v & 1 \end{pmatrix}\begin{pmatrix} p_k \\ q_k \end{pmatrix}, \qquad 1 \leqslant k \leqslant N \tag{3.4}$$

and introduce the Bargmann constraint [19]

$$\mathcal{P}g_0 = \sum_{k=1}^{N} \nabla\lambda_k, \tag{3.5}$$

which can be written as

$$\frac{v}{1+uv} = \langle \Lambda Eq, Eq\rangle, \qquad \frac{u}{1+uv} = -\langle p, p\rangle, \tag{3.6}$$

where $\langle ., .\rangle$ stands for the canonical inner product in $\mathcal{R}^N$, $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_N)$, $p = (p_1, \ldots, p_N)^T$, $q = (q_1, \ldots, q_N)^T$. Utilizing (3.6) and (3.4), we have a polynomial

$$(\langle Ap, p\rangle - \langle p, p\rangle)v^2 + (2\langle Ap, q\rangle - 1)v + \langle Aq, q\rangle = 0, \tag{3.7}$$

where $A = \mathrm{diag}\left(\frac{\lambda_1}{1+\lambda_1}, \ldots, \frac{\lambda_N}{1+\lambda_N}\right)$. Resorting to (3.7) and the second expression of (3.6), we obtain

$$\begin{aligned} u &= -\frac{\langle p, p\rangle(\langle Ap, p\rangle - \langle p, p\rangle)}{\langle Ap, p\rangle - \frac{1}{2}\langle p, p\rangle - \langle p, p\rangle\langle Ap, q\rangle + h_0\langle p, p\rangle}, \\ v &= \frac{1}{\langle Ap, p\rangle - \langle p, p\rangle}\left(\frac{1}{2} - \langle Ap, q\rangle + h_0\right) \end{aligned} \tag{3.8}$$

with

$$h_0^2 = \left(\langle Ap, q\rangle - \frac{1}{2}\right)^2 + \langle Aq, q\rangle(\langle p, p\rangle - \langle Ap, p\rangle). \tag{3.9}$$

Equation (3.8) can be written as

$$(u, v)^T = f(p, q), \tag{3.10}$$

where $f(p, q) = (f_1(p, q), f_2(p, q))$ is defined by the right side of (3.8). Substituting (3.8) into (3.4) and writing it in a vector form, we arrive at

$$E\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} B + (1 + f_1(p,q)f_2(p,q))B\Lambda & f_1(p,q)B\Lambda \\ f_2(p,q)B & B \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix} = \varphi\begin{pmatrix} p \\ q \end{pmatrix} \tag{3.11}$$

with $B = \mathrm{diag}\left(\frac{1}{\sqrt{1+\lambda_1}}, \ldots, \frac{1}{\sqrt{1+\lambda_N}}\right)$, $B^2 = I - A$.

Proposition 3.1. *φ is a symplectic map in $(\mathcal{R}^{2N}, \mathrm{d}p \wedge \mathrm{d}q)$.*

Proof. Through direct calculations. □

In order to prove the integrability of φ, we define the Poisson bracket of two functions in the symplectic space $(\mathcal{R}^{2N}, \mathrm{d}p \wedge \mathrm{d}q)$

$$\{h_1, h_2\} = \sum_{k=1}^{N} \left(\frac{\partial h_1}{\partial q_k}\frac{\partial h_2}{\partial p_k} - \frac{\partial h_1}{\partial p_k}\frac{\partial h_2}{\partial q_k} \right)$$

and a bilinear function $Q_\lambda(\xi, \eta)$ on $\mathcal{R}^N$

$$Q_\lambda(\xi, \eta) = \langle (\lambda I - \Lambda)^{-1}\xi, \eta\rangle = \sum_{i=1}^{N} \frac{\xi_i \eta_i}{\lambda - \lambda_i}.$$

We introduce

$$G_\lambda = \sum_{k=1}^{N} \frac{\widetilde{\nabla}\lambda_k}{\lambda - \lambda_k} + \frac{1}{\lambda}\delta_0 = \frac{1}{\lambda}\begin{pmatrix} \lambda Q_\lambda(\Lambda E q, E q) \\ -\lambda Q_\lambda(p, p) \\ Q_\lambda(\Lambda p, q) + \frac{1}{2} \end{pmatrix}, \tag{3.12}$$

with $\delta_0 = \left(0, 0, \frac{1}{2} - \langle p, q\rangle\right)^T$, which satisfies

$$(K - \lambda J)G_\lambda = 0 \tag{3.13}$$

with the aid of (3.2). Based on proposition 2.1, we define a Lax matrix V_λ by

$$V_\lambda \equiv \sigma[G_\lambda] = \begin{pmatrix} \frac{1}{2} + Q_\lambda(\Lambda p, q) & -\lambda Q_\lambda(p, p) \\ Q_\lambda(\Lambda q, q) & -\frac{1}{2} - Q_\lambda(\Lambda p, q) \end{pmatrix}, \tag{3.14}$$

that satisfies the stationary discrete zero-curvature equation

$$(E V_\lambda)U - U V_\lambda = 0. \tag{3.15}$$

According to corollary 2.2, $F_\lambda = \det V_\lambda$ is invariant under the action of the symplectic map φ, and yields the generating function of integrals of motion as follows:

$$F_\lambda = -\frac{1}{4} - Q_\lambda(\Lambda p, q) - Q_\lambda(\Lambda p, q)^2 + \lambda Q_\lambda(p, p) Q_\lambda(\Lambda q, q) = -\frac{1}{4} + \sum_{m=0}^{\infty} \frac{1}{\lambda^{m+1}} F_m \tag{3.16}$$

with

$$F_0 = -\langle \Lambda p, q\rangle + \langle p, p\rangle\langle \Lambda q, q\rangle, \tag{3.17}$$

$$F_1 = -\langle \Lambda^2 p, q\rangle - \langle \Lambda p, q\rangle^2 + \langle p, p\rangle\langle \Lambda^2 q, q\rangle + \langle \Lambda p, p\rangle\langle \Lambda q, q\rangle, \tag{3.18}$$

$$F_m = -\langle \Lambda^{m+1} p, q\rangle + \langle p, p\rangle\langle \Lambda^{m+1} q, q\rangle + \sum_{j=0}^{m-1} \begin{vmatrix} \langle \Lambda^{j+1} p, p\rangle & \langle \Lambda^{m-j} p, q\rangle \\ \langle \Lambda^{j+1} p, q\rangle & \langle \Lambda^{m-j} q, q\rangle \end{vmatrix}. \tag{3.19}$$

Here the expansion, $Q_\lambda(\xi, \eta) = \sum_{m=0}^{\infty} \lambda^{-m-1}\langle \Lambda^m \xi, \eta\rangle$, is used.

For the sake of convenience in a series of later calculations, we view F_λ as Hamiltonian in the symplectic space $(\mathcal{R}^{2N}, \mathrm{d}p \wedge \mathrm{d}q)$ and denote the variable of F_λ flow by τ_λ. A direct calculation gives the canonical equations of F_λ flow:

$$\frac{\mathrm{d}}{\mathrm{d}\tau_\lambda}\begin{pmatrix} p_k \\ q_k \end{pmatrix} = I\nabla_k F_\lambda = \begin{pmatrix} -\frac{\partial F_\lambda}{\partial q_k} \\ \frac{\partial F_\lambda}{\partial p_k} \end{pmatrix} = W(\lambda, \lambda_k)\begin{pmatrix} p_k \\ q_k \end{pmatrix}, \tag{3.20}$$

where

$$W(\lambda,\mu)=\frac{2\lambda}{\lambda-\mu}V_\lambda+2\begin{pmatrix}-V_\lambda^{(11)} & -V_\lambda^{(12)}\\ 0 & V_\lambda^{(11)}\end{pmatrix}. \tag{3.21}$$

Therefore, a direct calculation shows that the following assertion holds.

Proposition 3.2. *The matrix V_μ satisfies the Lax equation*

$$\frac{\mathrm{d}}{\mathrm{d}\tau_\lambda}V_\mu=[W(\lambda,\mu),V_\mu]. \tag{3.22}$$

Corollary 3.3.

$$\{F_\mu,F_\lambda\}=0,\qquad \forall\lambda,\mu\in C, \tag{3.23}$$

$$\{F_k,F_j\}=0,\qquad \forall j,k=0,1,2\ldots. \tag{3.24}$$

Proof. Equation (3.22) implies that $F_\mu=\det V_\mu$ is invariant along the τ_λ flow. And the derivative of the function F_μ along the F_λ flow is exactly the Poisson bracket, that is

$$\{F_\mu,F_\lambda\}=\frac{\mathrm{d}}{\mathrm{d}\tau_\lambda}F_\mu=0.$$

Substituting expansion (3.16) into the above expression, we obtain by comparing the same power of λ and μ that $\{F_k,F_j\}=0,\forall j,k=0,1,2\ldots.$ □

4. Decomposition of discrete soliton equations

It is easy to see that (3.5) can be written as $g_0=\sum_{j=1}^N\widetilde{\nabla}\lambda_j+\delta_0$. Operating with $J^{-1}K$ upon this expression k times and noting $\ker J=\{cg_0|\forall c\}$ give

$$\sum_{j=1}^N{\lambda_j}^k\widetilde{\nabla}\lambda_j=g_k+c_1g_{k-1}+\cdots+c_kg_0,\qquad (k\geqslant 1) \tag{4.1}$$

in view of (2.2) and (3.2), where c_k is a constant of motion.

Proposition 4.1. *The solution $(p(n),q(n))^T=\varphi^n(p_0,q_0)^T$ of the discrete flow generated by the symplectic map φ is mapped by f into a solution of the stationary discrete nonlinear equation*

$$X_N+C_{N1}X_{N-1}+\cdots+C_{NN}X_0=0. \tag{4.2}$$

Proof. Define a polynomial

$$a(\lambda)=\prod_{j=1}^N(\lambda-\lambda_j)=\sum_{k=0}^N a_{N-k}\lambda^k. \tag{4.3}$$

Multiplied by a_{N-k} and summed with respect to k from 0 to N, we arrive at

$$0=\sum_{j=1}^N a(\lambda_j)\widetilde{\nabla}\lambda_j=g_N+C_{N1}g_{N-1}+\cdots+C_{NN}g_0, \tag{4.4}$$

where

$$C_{N(N-k)}=a_{N-k}+\sum_{l=1}^{N-k}a_{N-k-l}c_l.$$

Operating with $\mathcal{P}K$ on (4.4) yields (4.2). □

Proposition 4.2. *Under constraint (3.5), functions G_λ and g_λ have a direct relation*

$$G_\lambda = c_\lambda g_\lambda \tag{4.5}$$

with

$$c_\lambda = 1 + \sum_{k=1}^{\infty} c_k \lambda^{-k}.$$

Proof. Using (3.12) and (4.1), we have

$$\begin{aligned} G_\lambda &= \sum_{j=1}^{N} \frac{\widetilde{\nabla}\lambda_j}{\lambda - \lambda_j} + \frac{1}{\lambda}\delta_0 \\ &= \sum_{k=1}^{\infty} \frac{1}{\lambda^{k+1}}(g_k + c_1 g_{k-1} + \cdots + c_k g_0) + \lambda^{-1} g_0 = c_\lambda g_\lambda. \end{aligned}$$

□

As a corollary, we get the expression of the Lax matrix V_λ and the generating function F_λ of conserved integrals :

$$V_\lambda = \sigma(\lambda)[G_\lambda] = \sigma(\lambda)[c_\lambda g_\lambda], \tag{4.6}$$

$$F_\lambda = {c_\lambda}^2 \det \sigma[g_\lambda] = -\tfrac{1}{4}{c_\lambda}^2. \tag{4.7}$$

Let us introduce the generating function H_λ by $c_\lambda = 1 - 4H_\lambda$. Then we have

$$(1 - 4H_\lambda)^2 = -4F_\lambda, \tag{4.8}$$

which implies that

$$\begin{aligned} &H_0 = \frac{1}{2}F_0, \qquad H_1 = \frac{1}{2}\left(F_1 + {F_0}^2\right), \\ &H_m = \frac{1}{2}F_m + 2\sum_{\substack{j+k=m-1\\ j,k\geqslant 0}} H_j H_k, \qquad m \geqslant 1 \end{aligned} \tag{4.9}$$

with the help of the expansion of the generating function

$$H_\lambda = \sum_{m=0}^{\infty} H_m \lambda^{-m-1}. \tag{4.10}$$

Proposition 4.3. *The functions $\{H_m\}$, $m \geqslant 0$, are in involution in pairs, $\{H_m, H_l\} = 0$, for any $m, l \geqslant 0$.*

Proof. The involutivity of $\{H_k\}$ is based on the equalities

$$\{F_\mu, F_\lambda\} = \tfrac{1}{16}\{(1 - 4H_\lambda)^2, \qquad (1 - 4H_\mu)^2\} = 16\sqrt{F_\mu F_\lambda}\{H_\mu, H_\lambda\}$$

and

$$\{H_\mu, H_\lambda\} = \frac{1}{16\sqrt{F_\lambda F_\mu}}\{F_\mu, F_\lambda\} = 0.$$

□

Denote the variables of H_λ flow and H_m flow by t_λ and t_m, respectively. By the Leibnitz rule of the Poisson bracket, we get

$$\{\psi, H_\lambda\} = \frac{1}{2c_\lambda}\{\psi, F_\lambda\} \tag{4.11}$$

for any smooth function ψ due to (4.8), which implies that

$$\frac{\mathrm{d}}{\mathrm{d}t_\lambda} = \frac{1}{2c_\lambda}\frac{\mathrm{d}}{\mathrm{d}\tau_\lambda}. \tag{4.12}$$

Utilizing (3.6), (3.20) and the third expression of (3.1), we arrive at

$$\begin{aligned}
&\langle \Lambda Eq, Eq_{\tau_\lambda}\rangle = -Q_\lambda(\Lambda^2 Eq, Eq) + 2\langle \Lambda Eq, Eq\rangle Q_\lambda(\Lambda Ep, Eq),\\
&\langle p, p_{\tau_\lambda}\rangle = Q_\lambda(\Lambda p, p) - 2\langle p, p\rangle Q_\lambda(\Lambda p, q),\\
&-uQ_\lambda(\Lambda Eq, Eq) - vQ_\lambda(p, p) + \lambda^{-1}\Delta Q_\lambda(\Lambda p, q) = 0
\end{aligned} \tag{4.13}$$

and

$$\begin{aligned}
u_{\tau_\lambda} &= \frac{2(1+uv)}{1-uv}\left(u^2\langle \Lambda Eq, Eq_{\tau_\lambda}\rangle - \langle p, p_{\tau_\lambda}\rangle\right),\\
v_{\tau_\lambda} &= \frac{2(1+uv)}{1-uv}\left(\langle \Lambda Eq, Eq_{\tau_\lambda}\rangle - v^2\langle p, p_{\tau_\lambda}\rangle\right).
\end{aligned} \tag{4.14}$$

Substituting the first two expressions of (4.13) into (4.14) and using the third expression of (4.13), we obtain

$$\begin{aligned}
\frac{\mathrm{d}}{d_{\tau_\lambda}}\begin{pmatrix} u \\ v \end{pmatrix} &= \begin{pmatrix} -2\lambda(1+uv)Q_\lambda(p, p) - 2u(1+E)\left(\frac{1}{2} + Q_\lambda(\Lambda p, q)\right) \\ -2\lambda(1+uv)Q_\lambda(\Lambda Eq, Eq) + 2v(1+E)\left(\frac{1}{2} + Q_\lambda(\Lambda p, q)\right) \end{pmatrix}\\
&= 2\lambda \mathcal{P} J G_\lambda,
\end{aligned} \tag{4.15}$$

which gives rise to

$$\frac{\mathrm{d}}{d_{t_\lambda}}\begin{pmatrix} u \\ v \end{pmatrix} = \lambda \mathcal{P} J g_\lambda \tag{4.16}$$

in view of (4.12). Hence we get the following assertion.

Proposition 4.4. *Let $(p(n, t_m), q(n, t_m))^T$ be a compatible solution of the discrete φ flow (3.11) and H_m flow*

$$\frac{\partial}{\partial t_m}\begin{pmatrix} p \\ q \end{pmatrix} = I\nabla H_m = \begin{pmatrix} \frac{-\partial H_m}{\partial q} \\ \frac{\partial H_m}{\partial p} \end{pmatrix}. \tag{4.17}$$

Then $(u(n, t_m), v(n, t_m))^T = f(p(n, t_m), q(n, t_m))$ solves equation (2.6).

Especially, for $m = 0$, (4.17) is reduced to

$$\begin{aligned}
p_{t_0} &= -\frac{\partial H_0}{\partial q} = \frac{1}{2}\Lambda p - \langle p, p\rangle \Lambda q,\\
q_{t_0} &= \frac{\partial H_0}{\partial p} = -\frac{1}{2}\Lambda q + \langle \Lambda q, q\rangle p
\end{aligned} \tag{4.18}$$

with

$$H_0 = -\frac{1}{2}\langle \Lambda p, q\rangle + \frac{1}{2}\langle p, p\rangle\langle \Lambda q, q\rangle. \tag{4.19}$$

5. Elliptic coordinates and functional independence

It is easy to see that one of F_λ, $V_\lambda^{(12)}$, $V_\lambda^{(21)}$, as a rational function of λ, has simple poles at $\lambda_j' s$, since the coefficients of $(\lambda-\lambda_j)^2$ is zero in F_λ

$$F_\lambda = \lambda Q_\lambda(p,p)Q_\lambda(\Lambda q,q) - \left(\frac{1}{2} + Q_\lambda(\Lambda p,q)\right)^2 = -\frac{b(\lambda)}{4a(\lambda)} = -\frac{R(\lambda)}{4a(\lambda)^2}, \tag{5.1}$$

$$V_\lambda^{(12)} = -\lambda Q_\lambda(p,p) = -\langle p,p\rangle \lambda \frac{m(\lambda)}{a(\lambda)}, \tag{5.2}$$

$$V_\lambda^{(21)} = Q_\lambda(\Lambda q,q) = \langle \Lambda q,q\rangle \frac{n(\lambda)}{a(\lambda)}, \tag{5.3}$$

with

$$a(\lambda)=\prod_{k=1}^{N}(\lambda-\lambda_k), \qquad b(\lambda)=\prod_{k=1}^{N}(\lambda-\lambda_{N+k}), \qquad m(\lambda)=\prod_{k=1}^{N-1}(\lambda-\mu_k),$$
$$n(\lambda)=\prod_{k=1}^{N-1}(\lambda-\nu_k), \qquad R(\lambda)=a(\lambda)b(\lambda)=\prod_{k=1}^{2N}(\lambda-\lambda_k),$$

where μ_k and ν_k are called elliptic coordinates. By comparing the coefficients of λ^{-m} in the expansions of (5.2) and (5.3), we have

$$\frac{\langle \Lambda p,p\rangle}{\langle p,p\rangle} = \sum_{k=1}^{N}\lambda_k - \sum_{k=1}^{N-1}\mu_k, \tag{5.4}$$

$$\frac{\langle \Lambda^2 q,q\rangle}{\langle \Lambda q,q\rangle} = \sum_{k=1}^{N}\lambda_k - \sum_{k=1}^{N-1}\nu_k. \tag{5.5}$$

Using (4.18), (5.4) and (5.5), we arrive at

$$\begin{aligned}\partial_{t_0}\ln\langle p,p\rangle &= \sum_{k=1}^{N}\lambda_k - \sum_{k=1}^{N-1}\mu_k - 2\langle \Lambda p,q\rangle,\\ \partial_{t_0}\ln\langle \Lambda q,q\rangle &= \sum_{k=1}^{N-1}\nu_k - \sum_{k=1}^{N}\lambda_k + 2\langle \Lambda p,q\rangle,\end{aligned} \tag{5.6}$$

which, together with (4.19), imply

$$\partial_{t_0}\ln(\langle \Lambda p,q\rangle + \kappa_0) = \sum_{k=1}^{N-1}(\nu_k-\mu_k), \tag{5.7}$$

where $\kappa_0 = 2H_0$. Substituting (3.6) into (5.6) yields

$$\begin{aligned}\partial_{t_0}\ln\frac{u}{1+uv} &= \sum_{k=1}^{N-1}\mu_k - \sum_{k=1}^{N}\lambda_k + 2\langle \Lambda p,q\rangle,\\ \partial_{t_0}\ln\frac{v}{1+uv} &= \sum_{k=1}^{N-1}E\nu_k - \sum_{k=1}^{N}\lambda_k + 2E\langle \Lambda p,q\rangle.\end{aligned} \tag{5.8}$$

The calculation of the evolution of the elliptic coordinates along the F_λ flow is based on the components of the Lax equation (3.22):

$$\frac{\mathrm{d}V_\mu{}^{(12)}}{\mathrm{d}\tau_\lambda} = \frac{4\mu V_\lambda{}^{(11)}V_\mu{}^{(12)} - 4\mu V_\lambda{}^{(12)}V_\mu{}^{(11)}}{\lambda-\mu}, \tag{5.9}$$

$$\frac{\mathrm{d}V_\mu{}^{(21)}}{\mathrm{d}\tau_\lambda} = \frac{4\lambda V_\lambda{}^{(21)}V_\mu{}^{(11)} - 4\mu V_\lambda{}^{(11)}V_\mu{}^{(21)}}{\lambda-\mu}. \tag{5.10}$$

According to (5.1), we arrive at

$$V_{\mu_k}^{(11)} = \frac{\sqrt{R(\mu_k)}}{2a(\mu_k)}, \qquad V_{\nu_k}^{(11)} = \frac{\sqrt{R(\nu_k)}}{2a(\nu_k)},$$

$$\frac{1}{2\sqrt{R(\mu_k)}}\frac{\mathrm{d}\mu_k}{\mathrm{d}\tau_\lambda} = -\frac{\lambda m(\lambda)}{(\lambda-\mu_k)a(\lambda)m'(\mu_k)}, \tag{5.11}$$

$$\frac{1}{2\sqrt{R(\nu_k)}}\frac{\mathrm{d}\nu_k}{\mathrm{d}\tau_\lambda} = \frac{\lambda n(\lambda)}{(\lambda-\nu_k)a(\lambda)n'(\nu_k)}. \tag{5.12}$$

Resorting to the interpolation formula, with degrees not more than $g = N-1$, we have $(j = 1, 2, \ldots, N-1)$,

$$\sum_{k=1}^{g}\frac{\mu_k{}^{g-j}}{2\sqrt{R(\mu_k)}}\frac{\mathrm{d}\mu_k}{\mathrm{d}\tau_\lambda} = -\sum_{k=1}^{g}\frac{\lambda m(\lambda)\mu_k{}^{g-j}}{a(\lambda)(\lambda-\mu_k)m'(\mu_k)} = -\frac{\lambda^{g-j+1}}{a(\lambda)}, \tag{5.13}$$

$$\sum_{k=1}^{g}\frac{\nu_k{}^{g-j}}{2\sqrt{R(\nu_k)}}\frac{\mathrm{d}\nu_k}{\mathrm{d}\tau_\lambda} = \sum_{k=1}^{g}\frac{\lambda n(\lambda)\nu_k{}^{g-j}}{a(\lambda)(\lambda-\nu_k)n'(\nu_k)} = \frac{\lambda^{g-j+1}}{a(\lambda)}. \tag{5.14}$$

These formulae lead naturally to the consideration of the elliptic curve Γ given by the affine equation

$$\xi^2 - \tfrac{1}{4}R(\lambda) = 0.$$

The genus is $N-1$ since $\deg R = 2N$. Denote $P(\lambda) = \left(\lambda, \xi = \frac{1}{2}\sqrt{R(\lambda)}\right)$. The usual holomorphic differentials on Γ

$$\widetilde{\omega}_j = \frac{\lambda^{g-j}}{2\sqrt{R(\lambda)}}\,\mathrm{d}\lambda, \qquad 1 \leqslant j \leqslant N-1, \tag{5.15}$$

imply the introduction of the quasi-Abel–Jacobi coordinates

$$\widetilde{\phi}_j = \sum_{P\in D_1}\int_{P_0}^{P}\widetilde{\omega}_j, \qquad \widetilde{\psi}_j = \sum_{P\in D_2}\int_{P_0}^{P}\widetilde{\omega}_j, \qquad 1 \leqslant j \leqslant g. \tag{5.16}$$

with fixed point $P_0 \in \Gamma$, and the divisor

$$D_1 = \sum_{j=1}^{g} P(\mu_j), \qquad D_2 = \sum_{j=1}^{g} P(\nu_j).$$

Then (5.13) and (5.14) are written as

$$\frac{\mathrm{d}\widetilde{\phi}_j}{\mathrm{d}\tau_\lambda} = -\frac{\lambda^{N-j}}{a(\lambda)}, \qquad \frac{\mathrm{d}\widetilde{\psi}_j}{\mathrm{d}\tau_\lambda} = \frac{\lambda^{N-j}}{a(\lambda)}. \tag{5.17}$$

It is easy to see that coefficients in expansion [27]

$$\frac{\lambda^N}{a(\lambda)} = \frac{1}{(1-\lambda_1\lambda^{-1})\cdots(1-\lambda_N\lambda^{-1})} = 1 + \sum_{j=1}^{\infty} A_j \lambda^{-j} \tag{5.18}$$

can be determined recursively by

$$A_0 = 1, \qquad A_1 = s_1, \qquad A_k = \frac{1}{k}\left(s_k + \sum_{i+j=k, i,j\geqslant 1} s_i A_j\right), \tag{5.19}$$

where $s_l = \sum_{k=1}^{N} {\lambda_k}^l$. Comparing the coefficients of λ^{-m} in the expansion $\frac{\mathrm{d}\widetilde{\phi}_s}{\mathrm{d}\tau_\lambda}$ yields

$$\frac{\mathrm{d}\widetilde{\phi}_s}{\mathrm{d}\tau_m} = \{\widetilde{\phi}_s, F_m\} = -A_{m-s+1}, \qquad 1 \leqslant s \leqslant N-1, \qquad m \geqslant 0,$$

with the supplementary definition $A_{-k} = 0$, $(k = 1, 2, \ldots)$. Here τ_k stand for the variable of the F_k flow. Therefore, we obtain

$$\left(\frac{\mathrm{d}\widetilde{\phi}}{\mathrm{d}\tau_0}, \frac{\mathrm{d}\widetilde{\phi}}{\mathrm{d}\tau_1}, \ldots, \frac{\mathrm{d}\widetilde{\phi}}{\mathrm{d}\tau_{N-2}}\right) = -\begin{pmatrix} 1 & A_1 & A_2 & \cdots & A_{N-2} \\ & 1 & A_1 & \cdots & A_{N-3} \\ & & \ddots & \ddots & \vdots \\ & & & \ddots & A_1 \\ & & & & 1 \end{pmatrix}, \tag{5.20}$$

where $\widetilde{\phi} = (\widetilde{\phi}_1, \ldots, \widetilde{\phi}_{N-1})^T$.

Proposition 5.1. *(i)* $\{F_0, F_1, \ldots, F_{N-1}\}$ *are functionally independent. (ii)* $\{H_0, H_1, \ldots, H_{N-1}\}$ *are functionally independent.*

Proof. Recall the expression of the Poisson bracket by means of the symplectic structure $\omega^2 = \mathrm{d}p \wedge \mathrm{d}q : \{h_1, h_2\} = \omega^2(Idh_2, Idh_1)$. Assume that $\sum_{k=0}^{N-1} \gamma_k \,\mathrm{d}F_k = 0$. Then we have

$$\begin{aligned} 0 = \sum_{k=0}^{N-1} \gamma_k \,\mathrm{d}F_k(I\,\mathrm{d}\widetilde{\phi}_j) &= \sum_{k=0}^{N-1} \gamma_k \omega^2(I\,\mathrm{d}F_k, I\,\mathrm{d}\widetilde{\phi}_j) = \sum_{k=0}^{N-1} \gamma_k \{\widetilde{\phi}_j, F_k\} \\ &= \sum_{k=0}^{N-1} \gamma_k \frac{\mathrm{d}\widetilde{\phi}_j}{\mathrm{d}\tau_k} \qquad (1 \leqslant j \leqslant N-1), \end{aligned} \tag{5.21}$$

which implies $\gamma_0 = \cdots = \gamma_{N-1} = 0$ since the coefficient of the determinant is equal to 1 by (5.20). Hence $F_0, \ldots, F_{N-1}$ are functionally independent. By using (4.9), we obtain

$$\begin{pmatrix} \mathrm{d}F_o \\ \mathrm{d}F_1 \\ \vdots \\ \mathrm{d}F_{N-1} \end{pmatrix} = 2 \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ & 1 & 0 & \cdots & 0 \\ & & \ddots & \ddots & \vdots \\ & * & & \ddots & 0 \\ & & & & 1 \end{pmatrix} \begin{pmatrix} \mathrm{d}H_o \\ \mathrm{d}H_1 \\ \vdots \\ \mathrm{d}H_{N-1} \end{pmatrix}. \tag{5.22}$$

Thus $\mathrm{d}H_0, \mathrm{d}H_1, \ldots, \mathrm{d}H_{N-1}$ are also linearly independent. □

Noting that F_λ is invariant under the action of the symplectic map (3.11), which means that F_m is also invariant. Noting (3.24), we have the assertions [22, 28, 29].

Theorem 5.2. *The symplectic map φ defined by (3.11) is completely integrable in the Liouville sense.*

Theorem 5.3. *The systems $(\mathcal{R}^{2N}, \mathrm{d}p \wedge \mathrm{d}q, F_m)$ and $(\mathcal{R}^{2N}, \mathrm{d}p \wedge \mathrm{d}q, H_m)$ are completely integrable in the Liouville sense.*

6. Straightening out of the continuous flow

Let $a_1, b_1, \ldots, a_g, b_g$, be the canonical basis of cycles on the hyperelliptic curve Γ, and

$$C = (A_{jk})^{-1}_{g\times g}, \qquad A_{jk} = \int_{a_k} \widetilde{\omega}_j. \tag{6.1}$$

Then we have the normalized holomorphic differential by

$$\omega_s = \sum_{j=1}^{g} C_{ij}\widetilde{\omega}_j, \qquad \omega = C\widetilde{\omega} = (\omega_1, \ldots, \omega_g)^T \tag{6.2}$$

with the properties

$$\int_{a_k} \omega_s = \delta_{sk}, \qquad \int_{b_k} \omega_s = B_{sk}, \tag{6.3}$$

where the matrix $B = (B_{sk})$ is symmetric with positively definite imaginary part and is used to construct the Riemann-theta function of Γ [30, 31]:

$$\theta(\zeta) = \sum_{z\epsilon z^g} \exp \pi\sqrt{-1}(\langle Bz, z\rangle + 2\langle \zeta, z\rangle), \qquad \zeta\epsilon C^g. \tag{6.4}$$

The Abel map $\mathcal{A}$: $\mathrm{Div}(\Gamma) \longrightarrow \mathcal{J}(\Gamma) = C^g/\mathcal{T}$ is defined by

$$\mathcal{A}(P) = \int_{P_0}^{P} \omega, \qquad \mathcal{A}\left(\sum n_k p_k\right) = \sum n_k \mathcal{A}(p_k), \tag{6.5}$$

where $\mathrm{Div}(\Gamma)$ is the divisor group and the lattice $\mathcal{T}$ is spanned by the periodic vectors $\{\delta_j, B_j\}$ with components given by (6.3). We introduce the Abel–Jacobi coordinates

$$\phi = \mathcal{A}(D_1) = C\widetilde{\phi}, \qquad \psi = \mathcal{A}(D_2) = C\widetilde{\psi}. \tag{6.6}$$

Assume that $s_k = \lambda_1^k + \cdots + \lambda_{2g+2}^k$, then the coefficients in [27]

$$\frac{\lambda^{g+1}}{\sqrt{R(\lambda)}} = \sum_{k=0}^{\infty} \Lambda_k \lambda^{-k} \tag{6.7}$$

satisfies the recursive formula

$$\Lambda_0 = 1, \qquad \Lambda_1 = \frac{1}{2}s_1, \qquad \Lambda_k = \frac{1}{2k}\left(s_k + \sum_{i+j=k, i,j\geqslant 1} s_i\Lambda_j\right). \tag{6.8}$$

Let $C_1, C_2, \ldots, C_g$ be the column vectors of C. Then we have the expansion

$$\frac{\lambda^{g+1}}{2\sqrt{R(\lambda)}}(C_1\lambda^{-1} + \cdots + C_g\lambda^{-g}) = \sum_{k=0}^{\infty} \Omega_k \lambda^{-k-1}, \tag{6.9}$$

where the coefficients are given by

$$\Omega_0 = \tfrac{1}{2}C_1, \qquad \Omega_k = \tfrac{1}{2}(\Lambda_k C_1 + \Lambda_{k-1}C_2 + \cdots + \Lambda_{k-g+1}C_g). \tag{6.10}$$

Theorem 6.1. *The H_k flow is straightened out by the Abel–Jacobi coordinates*

$$\frac{d\phi}{dt_k} = -\Omega_k, \qquad \frac{d\psi}{dt_k} = \Omega_k. \tag{6.11}$$

Proof. According to (5.17) and (6.6), we obtain

$$\frac{d\phi}{d\tau_\lambda} = C\frac{d\widetilde{\phi}}{d\tau_\lambda} = -\frac{\lambda^N}{a(\lambda)}(\lambda^{-1}C_1 + \cdots + \lambda^{-(N-1)}C_{N-1}).$$

Therefore, we have from (4.12) that

$$\begin{aligned}\frac{d\phi}{dt_\lambda} &= \frac{1}{2c_\lambda}\frac{d\phi}{d\tau_\lambda} = -\frac{\lambda^N}{2c_\lambda a(\lambda)}(\lambda^{-1}C_1 + \cdots + \lambda^{-(N-1)}C_{N-1})\\ &= -\frac{\lambda^N}{\sqrt{R(\lambda)}}(\lambda^{-1}C_1 + \cdots + \lambda^{-(N-1)}C_{N-1})\\ &= -\sum_{k=0}^{\infty}\Omega_k\lambda^{-k-1}\end{aligned} \tag{6.12}$$

in view of (4.7) and (5.1). In a similar way, we obtain

$$\frac{d\psi}{dt_\lambda} = \sum_{k=0}^{\infty}\Omega_k\lambda^{-k-1},$$

which, together with (6.12), implies (6.11). The proof is completed. □

7. Straightening out of the discrete flow

In this section, we shall derive the straightening out of the discrete flow. We start from the spectral problem (2.1) with $\chi(n) = (p(n), q(n))^T$. Assume that the fundamental solution matrix of (2.1) is as follows:

$$M(n) = (\chi^1(n), \chi^2(n)) = \begin{pmatrix} p^{(1)}(n) & p^{(2)}(n) \\ q^{(1)}(n) & q^{(2)}(n)\end{pmatrix}, \qquad M(0) = \begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix}, \tag{7.1}$$

which can be expressed as

$$M(n+1) = U(n)U(n-1)\cdots U(0). \tag{7.2}$$

By mathematical induction, it is easy to prove that

$$\begin{aligned} p^{(1)}(n) &= \frac{\prod_{j=0}^{n-1}(1+u(j)v(j))}{(\sqrt{1+\lambda})^n}\left[\lambda^n + \lambda^{n-1}\left(\sum_{j=0}^{n-1}\frac{1}{1+u(j)v(j)}\right.\right.\\ &\quad\left.\left. + \sum_{j=0}^{n-2}\frac{u(j+1)v(j+1)}{(1+u(j)v(j))(1+u(j+1)v(j+1))}\right) + \cdots\right],\\ p^{(2)}(n) &= \frac{u(0)\prod_{j=1}^{n-1}(1+u(j)v(j))}{(\sqrt{1+\lambda})^n}\left[\lambda^n + \lambda^{n-1}\left(\sum_{j=1}^{n-1}\frac{1}{1+u(j)v(j)}\right.\right.\\ &\quad + \sum_{j=1}^{n-2}\frac{u(j+1)v(j+1)}{(1+u(j)v(j))(1+u(j+1)v(j+1))}\\ &\quad\left.\left. + \frac{u(1)}{u(0)(1+u(1)v(1))}\right) + \cdots\right],\end{aligned} \tag{7.3}$$

$$
\begin{aligned}
q^{(1)}(n) = {} & \frac{v(n-1)\prod_{j=0}^{n-2}(1+u(j)v(j))}{(\sqrt{1+\lambda})^n}\Bigg[\lambda^{n-1}+\lambda^{n-2}\Bigg(\sum_{j=0}^{n-2}\frac{1}{1+u(j)v(j)} \\
& + \sum_{j=0}^{n-3}\frac{u(j+1)v(j)}{(1+u(j)v(j))(1+u(j+1)v(j+1))} \\
& + \frac{v(n-2)}{(1+u(n-2)v(n-2))v(n-1)}\Bigg)+\cdots\Bigg], \\
q^{(2)}(n) = {} & \frac{u(0)v(n-1)\prod_{j=1}^{n-2}(1+u(j)v(j))}{(\sqrt{1+\lambda})^n}\Bigg[\lambda^{n-1}+\lambda^{n-2}\Bigg(\sum_{j=1}^{n-2}\frac{1}{1+u(j)v(j)} \\
& + \sum_{j=1}^{n-3}\frac{u(j+1)v(j)}{(1+u(j)v(j))(1+u(j+1)v(j+1))} \\
& + \frac{u(1)}{u(0)(1+u(1)v(1))}+\frac{v(n-2)}{v(n-3)(1+u(n-2)v(n-2))}\Bigg)+\cdots\Bigg].
\end{aligned}
\tag{7.4}
$$

The discrete commutative equation (3.15) satisfied by the Lax matrix V_λ is the key to straighten out the discrete flow generated by the symplectic map φ. This means that the solution space of the linear equation (2.1), $E\chi = U\chi$, is invariant under the action of V_λ. Let ρ and χ be the eigenvalue and eigenfunction, respectively, of the linear operator V_λ in the solution space. Then they satisfy

$$E\chi = U\chi, \qquad V_\lambda\chi = \rho\chi. \tag{7.5}$$

It is easy to see that $\det(\rho - V_\lambda) = \rho^2 + F_\lambda = 0$. Therefore, there are two eigenvalues $\rho^\pm = \pm\rho$, and

$$\rho = \frac{\sqrt{R(\lambda)}}{2a(\lambda)} \tag{7.6}$$

in view of (5.1)

Proposition 7.1. *The eigenvalue ρ of the Lax matrix V_λ is the generating function of conserved integrals $\{H_m\}$ of the symplectic map φ.*

The eigenfunctions of the Lax matrix V_λ are called the Baker functions after some kind of normalization, which can be taken as

$$\chi^\pm(n) = \chi^{(1)}(n) + b^\pm\chi^{(2)}(n), \tag{7.7}$$

$$\hat\chi^\pm(n) = c^\pm\chi^{(1)}(n) + \chi^{(2)}(n), \tag{7.8}$$

with

$$b^\pm = \frac{\pm\rho - V_\lambda^{(11)}(0)}{V_\lambda^{(12)}(0)}, \qquad c^\pm = \frac{V_\lambda^{(11)}(0)\pm\rho}{V_\lambda^{(21)}(0)}. \tag{7.9}$$

Theorem 7.2. *Let $p^\pm(n,\lambda)$ and $q^\pm(n,\lambda)$ be the first component of the Baker function $\chi^\pm(n,\lambda)$, and the second one of the Baker function $\hat\chi^\pm(n,\lambda)$, respectively. Then*

$$p^+(n,\lambda)p^-(n,\lambda) = \frac{\langle p(n), p(n)\rangle}{\langle p(0), p(0)\rangle}\prod_{j=1}^{N-1}\frac{\lambda-\mu_j(n)}{\lambda-\mu_j(0)}, \tag{7.10}$$

$$q^{+}(n,\lambda)q^{-}(n,\lambda)=\frac{\langle\Lambda q(n),q(n)\rangle}{\langle\Lambda q(0),q(0)\rangle}\prod_{j=1}^{N-1}\frac{\lambda-\nu_j(n)}{\lambda-\nu_j(0)}. \tag{7.11}$$

Proof. Using (3.15) and (7.2), we have

$$V_\lambda M(n)=M(n)V_\lambda(0), \tag{7.12}$$

from which we derive (7.10) and (7.11). □

Proposition 7.3. *For $\lambda\longrightarrow\infty$, we have*

$$\begin{aligned}p^{+}(n,\lambda)&=(1+u(0)v(0))\prod_{j=1}^{n-1}(1+u(j)v(j))\frac{\lambda^n}{(\sqrt{1+\lambda})^n}[1+o(\lambda^{-1})],\\ p^{-}(n,\lambda)&=-\frac{\langle p(n),p(n)\rangle}{u(0)\prod_{j=1}^{n-1}(1+u(j)v(j))}\frac{(\sqrt{1+\lambda})^n}{\lambda^n}[1+o(\lambda^{-1})].\end{aligned} \tag{7.13}$$

$$\begin{aligned}q^{+}(n,\lambda)&=\frac{v(n-1)}{\langle\Lambda q(0),q(0)\rangle}\prod_{j=1}^{n-2}(1+u(j)v(j))\frac{\lambda^n}{(\sqrt{1+\lambda})^n}[1+o(\lambda^{-1})],\\ q^{-}(n,\lambda)&=\frac{\langle\Lambda q(n),q(n)\rangle}{v(n-1)\prod_{j=1}^{n-2}(1+u(j)v(j))}\frac{(\sqrt{1+\lambda})^n}{\lambda^n}[1+o(\lambda^{-1})].\end{aligned} \tag{7.14}$$

Proof. Noting (3.14) and the Laurent expansion of function, we have

$$\begin{aligned}&\rho=\frac{1}{2}(1-4H_\lambda)=\frac{1}{2}-2\sum_{m=0}^{\infty}H_m\lambda^{-m-1}=\frac{1}{2}+o(\lambda^{-1}),\\ &V_\lambda^{(11)}(0)=\frac{1}{2}+Q_\lambda(\Lambda p(0),q(0))=\frac{1}{2}+o(\lambda^{-1}),\\ &V_\lambda^{(12)}(0)=-\langle p(0),p(0)\rangle(1+o(\lambda^{-1})),\\ &V_\lambda^{(21)}(0)=\langle\Lambda q(0),q(0)\rangle\lambda^{-1}(1+o(\lambda^{-1})),\end{aligned}$$

which implies

$$b^{+}=\frac{\rho-V_\lambda^{(11)}(0)}{V_\lambda^{(12)}(0)}=-\frac{1}{\langle p(0),p(0)\rangle}o(\lambda^{-1}), \tag{7.15}$$

$$c^{+}=\frac{V_\lambda^{(11)}(0)+\rho}{V_\lambda^{(21)}(0)}=\frac{\lambda}{\langle\Lambda q(0),q(0)\rangle}(1+o(\lambda^{-1})). \tag{7.16}$$

Substituting (7.3), (7.4), (7.15) and (7.16) into the equations

$$\begin{aligned}&p^{+}(n)=p^{(1)}(n)+b^{+}p^{(2)}(n), &&q^{+}(n)=c^{+}q^{(1)}(n)+q^{(2)}(n),\\ &p^{-}(n)=p^{(1)}(n)+b^{-}p^{(2)}(n), &&q^{-}(n)=c^{-}q^{(1)}(n)+q^{(2)}(n),\end{aligned}$$

a direct calculation shows that (7.13) and (7.14) hold. □

Proposition 7.4. *The Baker function $p(n,P)$ is of the properties: (i) $N-1$ simple zeros at $\mu_1(n),\dots,\mu_{N-1}(n)$ and a pole of the n-order at ∞_2 on the upper sheet of Γ; (ii) $N-1$ simple zeros at $\mu_1(0),\dots,\mu_{N-1}(0)$ and a zero of the n-order at ∞_1 on the lower sheet of Γ.*

Proposition 7.5. *The Baker function $q(n, P)$ is of the properties: (i) $N-1$ simple zeros at $\nu_1(0), \ldots, \nu_{N-1}(0)$ and a pole of the n-order at ∞_2 on the upper sheet of Γ; (ii) $N-1$ simple zeros at $\nu_1(n), \ldots, \nu_{N-1}(n)$ and a zero of the n-order at ∞_1 on the lower sheet of Γ.*

Theorem 7.6. *The discrete flows are straightened out by the Abel–Jacobi coordinates*

$$\Delta\phi(n) = \phi(n+1) - \phi(n) = \Omega_\varphi \quad (\mathrm{mod}\, \mathcal{T}), \tag{7.17}$$

$$\Delta\psi(n) = \psi(n+1) - \psi(n) = \Omega_\varphi \quad (\mathrm{mod}\, \mathcal{T}), \tag{7.18}$$

where

$$\Omega_\varphi = \int_{\infty_1}^{\infty_2} \omega.$$

Proof. Consider the meromorphic differential on Γ

$$\omega_\varphi(n) = \left\{ \frac{\mathrm{d}}{\mathrm{d}\lambda} \ln p(n, P) \right\} \mathrm{d}\lambda, \tag{7.19}$$

which has poles at $\mu_j(n)$ and $\mu_j(0)$ with the residue 1 and -1, respectively, and poles at ∞_2 and ∞_1 with the residue $-n$ and n, respectively. Differential (7.19) can be expressed as a linear combination of the normalized differential ω_j of the first kind, the differential Ω of the second kind and the normal differential $\omega(P, Q)$ of the third kind with residue 1 and -1 at P and Q. These differentials have the properties

$$\int_{a_j} \omega(P, Q) = 0, \qquad \int_{b_j} \omega(P, Q) = 2\pi\sqrt{-1} \int_Q^P \omega_j. \tag{7.20}$$

Then (7.19) can be written as

$$\omega_\varphi(n) = \Omega + n\omega(\infty_1, \infty_2) + \sum_{j=1}^{N-1} \omega[\mu_j(n), \mu_j(0)] + \sum_{j=1}^{N-1} \gamma_j \omega_j, \tag{7.21}$$

where γ_j are some complex numbers. The integral of (7.19) along a_k gives $\gamma_k = 2\pi\sqrt{-1} n_k$, while the integral of (7.21) along b_k yields

$$\sum_{j=1}^{N-1} \int_{\mu_j(0)}^{\mu_j(n)} \omega_k = n \int_{\infty_1}^{\infty_2} \omega_k + \sum_{j=1}^{N-1} (\delta_{jk} m_j - n_j B_{jk}), \qquad 1 \leqslant k \leqslant N-1, \tag{7.22}$$

where n_j and m_j are certain integers. Therefore, (7.17) holds.

In a similar way, we consider the meromorphic differential on Γ

$$\hat{\omega}_\varphi(n) = \left\{ \frac{\mathrm{d}}{\mathrm{d}\lambda} \ln q(n, P) \right\} \mathrm{d}\lambda, \tag{7.23}$$

we can prove that

$$\sum_{j=1}^{N-1} \int_{\nu_j(0)}^{\nu_j(n)} \omega_k = n \int_{\infty_1}^{\infty_2} \omega_k + \sum_{j=1}^{N-1} (\delta_{jk} \hat{m}_j - \hat{n}_j B_{jk}), \qquad 1 \leqslant k \leqslant N-1. \tag{7.24}$$

This completes the proof of this theorem. □

Corollary 7.7.

$$\phi(n) - \phi(0) = n\Omega_\varphi \qquad (\mathrm{mod}\, \mathcal{T}), \tag{7.25}$$

$$\psi(n) - \psi(0) = n\Omega_\varphi \qquad (\mathrm{mod}\, \mathcal{T}). \tag{7.26}$$

Based on theorems 6.1, 7.6 and corollary 7.7, the compatible solutions of various flows under the Abel–Jacobi coordinates can be obtained simply through a linear superposition. Therefore, the discrete nonlinear evolution equations (2.6) ($m \geqslant 1$) and (2.7) possess the compatible solutions under the Abel–Jacobi coordinates:

$$\begin{aligned} &\phi(n, t_0, t_m) = n\Omega_\varphi - t_0\Omega_0 - t_m\Omega_m + \phi_0, \\ &\psi(n, t_0, t_m) = n\Omega_\varphi + t_0\Omega_0 + t_m\Omega_m + \psi_0, \qquad m \geqslant 1. \end{aligned} \tag{7.27}$$

8. Quasi-periodic solutions

In order to give explicit solutions of (2.6) in the original coordinate, the following steps should be completed: $(\phi, \psi) \longrightarrow (\mu_j, \nu_j) \longrightarrow (u(n), v(n))$. According to the Riemann theorem [30–32], there exist constant vectors M_1 and $M_2 \in \mathcal{C}^{N-1}$ such that $\theta(\mathcal{A}(P(\lambda)) - \phi - M_1)$ has exactly $N-1$ zeros at $\lambda = \mu_1, \ldots, \mu_{N-1}$ and $\theta(\mathcal{A}(P(\lambda)) - \psi - M_2)$ has exactly $N-1$ zeroes at $\lambda = \nu_1, \ldots, \nu_{N-1}$. And we have the inversion formula

$$\sum_{j=1}^{N-1} \mu_j = I_1(\Gamma) - \sum_{s=1,\lambda=\infty_s}^{2} \operatorname{Res} \lambda \, \mathrm{d} \ln \theta(\mathcal{A}(P(\lambda)) - \phi - M_1), \tag{8.1}$$

$$\sum_{j=1}^{N-1} \nu_j = I_1(\Gamma) - \sum_{s=1,\lambda=\infty_s}^{2} \operatorname{Res} \lambda \, \mathrm{d} \ln \theta(\mathcal{A}(P(\lambda)) - \psi - M_2) \tag{8.2}$$

with the constant

$$I_1(\Gamma) = \sum_{j=1}^{N-1} \int_{a_j} \lambda \omega_j.$$

For the same λ, there are two points on different sheets of the Riemann surface Γ. Under the local coordinate $z = \lambda^{-1}$, noting (6.9), (5.15) and (6.2), we have

$$\omega = C\tilde{\omega} = (-1)^{s+1} \sum_{k=0}^{\infty} \Omega_k z^k \, \mathrm{d}z, \tag{8.3}$$

$$\mathcal{A}(P(z^{-1})) = -\eta_s - (-1)^s \sum_{k=0}^{\infty} \frac{1}{k+1} \Omega_k z^{k+1} \tag{8.4}$$

with

$$\eta_s = \int_{\infty_s}^{P_0} \omega.$$

Since the theta function is an even function, we have

$$\begin{aligned} &\theta(\mathcal{A}(P(z^{-1})) - \phi - M_1) = \theta(\phi + M_1 + \eta_s) - z(-1)^s \frac{\partial}{\partial t_0} \theta(\phi + M_1 + \eta_s) + o(z^2), \\ &\theta(\mathcal{A}(P(z^{-1})) - \psi - M_2) = \theta(\psi + M_2 + \eta_s) + z(-1)^s \frac{\partial}{\partial t_0} \theta(\psi + M_2 + \eta_s) + o(z^2). \end{aligned} \tag{8.5}$$

From (8.1), (8.2) and (8.5), we arrive at

$$\sum_{j=1}^{N-1} \mu_j = I_1 + \frac{\partial}{\partial t_0} \ln \frac{\theta(n\Omega_\varphi - t_0\Omega_0 - t_m\Omega_m + \pi_2)}{\theta(n\Omega_\varphi - t_0\Omega_0 - t_m\Omega_m + \pi_1)}, \tag{8.6}$$

$$\sum_{j=1}^{N-1} \nu_j = I_1 + \frac{\partial}{\partial t_0} \ln \frac{\theta(n\Omega_\varphi + t_0\Omega_0 + t_m\Omega_m + \hat{\pi}_1)}{\theta(n\Omega_\varphi + t_0\Omega_0 + t_m\Omega_m + \hat{\pi}_2)} \tag{8.7}$$

in view of (7.27), where constants

$$\pi_s = \phi_0 + M_1 + \eta_s, \qquad \hat{\pi}_s = \psi_0 + M_2 + \eta_s, \qquad s = 1, 2.$$

Putting (8.6) and (8.7) into (5.7) and integrating once with respect to t_0 yield

$$\langle \Lambda p, q\rangle = \kappa_1 \frac{\theta(n\Omega_\varphi - t_0\Omega_0 - t_m\Omega_m + \pi_1)\theta(n\Omega_\varphi + t_0\Omega_0 + t_m\Omega_m + \hat{\pi}_1)}{\theta(n\Omega_\varphi - t_0\Omega_0 - t_m\Omega_m + \pi_2)\theta(n\Omega_\varphi + t_0\Omega_0 + t_m\Omega_m + \hat{\pi}_2)} - \kappa_0 \tag{8.8}$$

with a constant of integration κ_1. Substituting (8.6)–(8.8) into (5.8) and integrating once with respect to t_0, we obtain explicit solutions of the discrete nonlinear evolution equations (2.6) ($m \geqslant 1$) and (2.7) under the original coordinates $u(n)$ and $v(n)$. Especially, for $m = 1$, these explicit solutions solve the following (2+1)-dimensional nonlinear differential-difference equation ($t_0 = y$):

$$\begin{aligned} u(n)_t &= u(n)_{yy} - 2\Delta \frac{u(n)^2 v(n-1)}{(1+u(n)v(n))^2(1+u(n-1)v(n-1))}, \\ v(n)_t &= -v(n)_{yy} + 2\Delta^* \frac{u(n+1)v(n)^2}{(1+u(n)v(n))^2(1+u(n+1)v(n+1))}. \end{aligned} \tag{8.9}$$

In fact, from (2.7) we have

$$\Delta \frac{1}{1+u(n)v(n)} \Delta \frac{u(n)}{1+u(n)v(n)} = u(n)_{yy} + \Delta \frac{u(n)v(n)u(n)_y + u(n)^2 v(n)_y}{(1+u(n)v(n))^2} \tag{8.10}$$

and

$$\begin{aligned} \frac{u(n+1)}{1+u(n+1)v(n+1)} &= u(n)_y + \frac{u(n)}{1+u(n)v(n)}, \\ \frac{v(n-1)}{1+u(n-1)v(n-1)} &= -v(n)_y + \frac{v(n)}{1+u(n)v(n)}. \end{aligned} \tag{8.11}$$

Substituting (8.10) and (8.11) into (2.8) yields the first expression of (8.9). Similarly, we can prove the second expression of (8.9).

In what follows, we now discuss the continuous limit of (8.9). Noting (2.9), we obtain

$$\begin{aligned} \mathrm{i}u(n)_t &= \mathrm{i}u(n)_{yy} + 2(u(n)^2 v(n))_x h^3 + 2(u_x^2 v + uvu_{xx} - \tfrac{1}{2}u^2 v_{xx})h^4 + o(h^4), \\ v(n)_t &= -v(n)_{yy} - 2\mathrm{i}(v(n)^2 u(n))_x h^3 + 2\mathrm{i}(v_x^2 u + uvv_{xx} - \tfrac{1}{2}v^2 u_{xx})h^4 + o(h^4). \end{aligned}$$

Let $Y = h^2 y, T = \mathrm{i}h^4 t, X = hx$. Then we obtain a (2+1)-dimensional coupled DNLS equation by taking $h \to 0$

$$\mathrm{i}u_T - u_{YY} + 2\mathrm{i}(u^2 v)_X = 0, \qquad \mathrm{i}v_T + v_{YY} + 2\mathrm{i}(v^2 u)_X = 0, \tag{8.12}$$

which is reduced to the (2+1)-dimensional DNLS equation (1.7) as $u = v^*$.

Acknowledgments

Project 10471132 was supported by National Natural Science Foundation of China.

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